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Normal and Generalized Bose Condensation in Traps: One Dimensional Examples

András Sütő

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We prove the following results. (i) One-dimensional Bose gases which interact via unscaled integrable pair interactions and are confined in an external potential increasing faster than quadratically undergo a complete generalized Bose-Einstein condensation (BEC) at any temperature, in the sense that a macroscopic number of particles are distributed on a $o(N)$ number of one-particle states. (ii) In a one dimensional harmonic trap the replacement of the oscillator frequency ω by $\omega \ln N/N$ gives rise to a phase transition at $a \equiv \hbar\omega\beta = 1$ in the noninteracting gas. For $a < 1$ the limit distribution of n_0/N^a is exponential and $\langle n_0 \rangle / N^a \to 1$. For $a > 1$ there is BEC with a condensate density $\langle n_0 \rangle / N \to 1 - a^{-1}$. For $a \geqslant 1$, $(\ln N/N)(n_0 - \langle n_0 \rangle)$ is asymptotically distributed following Gumbel's law. For any $a > 0$ the free energy is $-(\pi^2/6\beta a)N/\ln N + o(N/\ln N)$, with no singularity at $a = 1$. (iii) In Model (ii) both above and below the critical temperature the gas undergoes a complete generalized BEC, thus providing a coexistence of ordinary and generalized condensates below the critical point. (iv) Adding an interaction $\langle U_N \rangle = o(N \ln N)$ to Model (ii) we prove that a complete generalized BEC occurs for any $\beta > 0$.

KEY WORDS: Trapped Bose gas; one dimension; generalised Bose-Einstein condensation; interactions.

1. INTRODUCTION

The idea of Bose-Einstein condensation (BEC) in the ground state of a one-dimensional interacting Bose gas first appeared in Girardeau's 1960 paper about the δ-gas in the impenetrable limit.(1) Based on an approximate calculation, Girardeau suggested that there is no BEC in the ground

Research Institute for Solid State Physics and Optics, Hungarian Academy of Sciences, P. O. Box 49, H-1525 Budapest, Hungary, e-mail: suto@szfki.hu

state but, instead, there is a complete generalized BEC (GBEC) in the sense that

$$\lim_{s\to 0}\lim_{N\to\infty}\frac{1}{N}\sum_{|k|<s\rho}\langle n_k\rangle = 1. \tag{1}$$

Here N is the number of particles, $\langle n_k\rangle$ is the ground state expectation value of the occupation number of the one-particle state $\sim \exp(ikx)$ and ρ is the density. Later Schultz(2) disproved this conjecture by showing that the double limit in (1) yields zero. A subsequent work by Lenard(3) on the momentum distribution confirmed this conclusion. Somewhat later the absence of BEC(4) (see ref. 5 for a rigorous proof) and that of GBEC(6) were shown in one- and two-dimensional homogenous systems at positive temperatures T. Bogoliubov's inequality(7) which these results are based on becomes trivial at $T=0$, and BEC or GBEC in the ground state of the *soft*- δ-gas has been a most interesting open problem ever since the Bethe-Ansatz solution by Lieb and Liniger was published.(8) Different approximate methods, as an effective long-range theory,(9) expansion to first order in $1/c$ where c is the prefactor of δ(10) and conformal field theoric approach,(11) predict an algebraic decay of the ground state expectation value of the boson field operator $a^*(x)a(0)$, incompatible with an off-diagonal long-range order. Also, some arguments(12,13) excluding the spontaneous breakdown of a continuous symmetry in the ground state of certain one-dimensional systems probably apply to the δ-gas. On the other hand, no conjecture seems to exist concerning the possibility of GBEC.

BEC can occur in one dimension if the gas is confined in an external potential. In ref. 14 it was shown that there is BEC at any finite temperature if the bosons are in a fixed (not N-dependent) external potential and interact via a suitably scaled pair interaction,

$$u_N(\mathbf{x}) = b_N u(\alpha_N \mathbf{x}). \tag{2}$$

Here u is an integrable positive function (a soft δ in one dimension is allowed) and b_N and α_N have to be chosen so that

$$\int u_N\,\mathrm{d}\mathbf{x} \leqslant C/N \tag{3}$$

for some finite C. This result is valid in any dimension $d\geqslant 1$ and imposes

$$b_N \leqslant C\alpha_N^d/N. \tag{4}$$

Interactions satisfying this condition cover in any dimension a whole range from mean-field types (α_N and, thus, b_N tending to zero as N goes to infinity) to sharply concentrated ones (α_N and b_N going to infinity). Even the mean-field type interactions are nontrivial if the confining potential is locally bounded, thus allowing particles to decrease their interaction by increasing their separation, or if $u(\mathbf{x})$ tends to infinity as $\mathbf{x}$ goes to zero.

For the noninteracting gas Bose condensation in a fixed trap is a pure ground state phenomenon. Let $H^0 = -\frac{\hbar^2}{2m}\Delta + V$ where V is chosen in such a way that $\operatorname{tr} e^{-\beta H^0} < \infty$ for any $\beta > 0$. (By Symanzik's version[15] of the Golden-Thompson-Symanzik inequality

$$\operatorname{tr} e^{-\beta H^0} \leqslant \lambda_\beta^{-d} \int e^{-\beta V(\mathbf{r})} \, d\mathbf{r} \tag{5}$$

where $\lambda_\beta = \hbar\sqrt{2\pi\beta/m}$, this holds if V increases faster than logarithmically.) As it was shown in ref. 14, at any temperature the $N \to \infty$ limit of the thermal equilibrium state of the corresponding noninteracting gas remains essentially a finite perturbation of the ground state. For this reason, at any finite temperature there is an asymptotically complete condensation into φ_0, the ground state of H^0. Because the ground state is common for the Bose and Boltzmann gases, there is BEC, although incomplete, at any temperature even in the trapped ideal quantum Boltzmann gas! The reduced one-particle density matrix of the latter is $Ne^{-\beta H^0}/\operatorname{tr} e^{-\beta H^0}$, whose largest eigenvalue $Ne^{-\beta\varepsilon_0}/\operatorname{tr} e^{-\beta H^0}$ gives the mean number of particles in φ_0 (ε_0 is the lowest eigenvalue of H^0).

Adding a scaled pair interaction (2) with property (4) to the noninteracting gas while keeping the trap fixed has a nontrivial effect, which can be expected by comparing energies: The total free energy of the ideal gas is of the order of 1 (apart from the trivial ground state energy $N\varepsilon_0$ which could be chosen to be zero), while the total interaction energy under condition (3) is of the order of N. BEC survives such a perturbation at all temperatures. Working directly at zero temperature, in three dimensions for $b_N = N^2$ and $\alpha_N = N$ (which is Gross-Pitaevskii scaling, satisfying (3)) Lieb and Seiringer were able to prove a complete BEC into the minimizer of the Gross-Pitaevskii energy functional[16]. This is very different from φ_0, also showing the nontrivial effect of the interaction.

The strength of an integrable interaction can be measured either by its integral or by its scattering length. Whether or not a pair interaction u_N whose integral is of order $1/N$ counts to be weak from the other point of view, depends on the space dimension. If $u_N(\mathbf{x}) = \alpha_N^2 u(\alpha_N \mathbf{x})$, the scattering length of u_N is $1/\alpha_N$ times the scattering length of u. In three dimensions (3) imposes $\alpha_N \sim N$, thus both the integral and the scattering length are of order $1/N$. In two dimensions $\int \alpha_N^2 u(\alpha_N \mathbf{x})\,\mathrm{d}\mathbf{x} = \int u(\mathbf{x})\,\mathrm{d}\mathbf{x}$, so u_N has to be chosen in a different form in order to comply with (3). In one dimension, with the necessary choice $\alpha_N \sim 1/N$, we get a scattering length $\sim N$, therefore the two criteria of strength contradict each other. This last example is a mean-field type interaction while e.g., $\alpha_N = N^2$ and $b_N = N$ represent the opposite limit in one dimension.

Further insight is obtained if scaling of the interaction is transformed into scaling of the potential and the temperature:

$$\sum_{i=1}^{N}[-\frac{\hbar^2}{2m}\Delta_{\mathbf{x}_i} + V(\mathbf{x}_i)] + \sum_{1\leqslant i<j\leqslant N} \alpha_N^2 u(\alpha_N(\mathbf{x}_i - \mathbf{x}_j))$$
$$= \alpha_N^2 \left\{ \sum_{i=1}^{N}[-\frac{\hbar^2}{2m}\Delta_{\mathbf{y}_i} + \alpha_N^{-2} V(\alpha_N^{-1}\mathbf{y}_i)] + \sum_{1\leqslant i<j\leqslant N} u(\mathbf{y}_i - \mathbf{y}_j) \right\} \quad (6)$$

where $\mathbf{y}_i = \alpha_N \mathbf{x}_i$. Because α_N^2 multiplies the inverse temperature β, we obtain a joint limit during which $N \to \infty$ and in three dimensions the temperature goes to zero and the trap opens in such a way that the particle density tends to zero, while in one dimension the temperature goes to infinity and the trap closes so that the density diverges. The three dimensional example suggests that whatever high the unscaled temperature, Gross-Pitaevskii scaling $\alpha_N = N$ may reduce the thermal equilibrium state to the ground state. It is at least true that BEC remains complete at positive temperatures (see ref. 17 for the Gross-Pitaevskii limit and the discussion of Section 2 for the general case (2)).

The analysis of the behaviour of a many-body system in a scaled external field or in the presence of scaled interactions comes about naturally in the study of trapped Bose gases. Taking the limit $N \to \infty$ and scaling certain quantities during this limit is an abstraction to obtain a mathematically clear-cut answer, not different in its philosophy from the thermodynamic limit in homogenous systems. The most frequently used scaling limit is related to the Gross-Pitaevskii energy functional[18,19] which describes the ground-state properties of dilute Bose gases. As mentioned earlier, in a three-dimensional trap this scaling is characterized by fixing Nl_N, where l_N is the scattering length of the pair interaction, see also ref. 16. Considering the scattering length as an effective diameter of the particle, space filling is seen to decrease as $1/N^2$ in this limit. Condition (2)–(4) is a generalization of the Gross-Pitaevskii scaling of the interaction. As a different example, scaling of the external confining potential appears in the theoretical discussion of BEC in elongated traps, realized experimentally more recently.[20,21] Various aspects of a one-dimensional behaviour can become manifest because a tight transverse trapping makes transversal degrees of freedom freeze out. Theoretically this can be achieved by applying different scalings of the confining potential in the transverse and longitudinal directions. These studies reveal a rich structure of low-dimensional regimes.[22–24] Note also that a theoretical work on one-dimensional bosons in a scaled harmonic trap was published well before the first experimental realization.[25]

The present paper is motivated by two naturally arising questions. What happens with BEC in dense or strongly interacting trapped gases? How does interaction modify BEC occurring with a phase transition in the ideal trapped gas? To study these questions, first (Section 2) for a gas in a fixed trap we relax the condition (3) at the price of not being able to prove BEC, only GBEC. We obtain the best result for one dimensional anharmonic and other superharmonic traps, i.e., potentials with a faster than quadratic increase. We can prove a complete GBEC at any finite temperature in the presence of unscaled pair interactions, that is, the total interaction energy increasing as N^2. This result holds also for attractive interactions, showing that the collapse of an attractive Bose gas can be considered as a GBEC. A box is a superharmonic trap and thus, in principle, our findings may have implications for the homogenous δ-gas. However, we can prove GBEC only if $c/\rho \to 0$ as $N \to \infty$. In other words, we just fail to prove it for the interacting gas. A reason of this failure may be that probably there is no Bose condensation, ordinary or generalized, in the ground state of the homogenous soft-δ-gas. Although our method is too weak to prove BEC in homogenous interacting gases, some nontrivial bounds can be obtained and are presented in Section 2 (Eqs. (29) and (30)).

In the case of a gas in a fixed trap our proof either yields BEC/GBEC for any finite temperature or does not yield it at all. In the second part (Section 3) of the paper we are interested in the stability against interactions of BEC which occurs through a phase transition in the noninteracting gas. Because $T_c = \infty$ for fixed traps, it is natural to look for a finite T_c by opening the trap together with $N \to \infty$. During this limit the separation of energy levels and, in particular, the gap above the ground state tend to zero, a situation similar to that occurring in the homogenous gas. As a consequence, the difficulties to include interactions start to resemble those appearing in homogenous systems. We consider only the harmonic potential, $V(x) = m\omega^2 x^2/2$. In this case the relevant dimensionless parameter is $a = \hbar\omega\beta$. In three dimensions it was found[18] that the N-dependent critical temperature of the noninteracting gas is given by $k_B T_c(N) = \hbar\omega[N/\zeta(3)]^{1/3}$ which is equivalent to say that the replacement of a by $aN^{-1/3}$ gives rise to BEC at $a_c = \zeta(3)^{1/3} \approx 1.063$. Although the one-particle density of states is different, this scaling qualitatively reproduces the properties of the homogenous gas. Since the interactions for which we can prove the survival of condensation are too weak in three dimensions ($o(N^{1/3})$, negligible on the scale N of the free energy of the noninteracting gas), in Section 3.1 we consider the analogous problem in one dimension. Earlier Ketterle and van Druten[25] found that in the noninteracting gas the N-dependent critical temperature was $k_B T_c(N) = \hbar\omega N/\ln(2N)$. We start with a detailed study of the noninteracting system. We replace ω by $\omega\gamma_N$ or a by $a\gamma_N$ where γ_N tends to zero and discuss the different possibilities. Depending on the decay rate of γ_N, T_c may be infinite or zero or may have a finite positive value. If $\gamma_N = \ln N/N$, T_c is finite positive with $a_c = 1$ (equivalently, $k_B T_c(N) = \hbar\omega N/\ln N$; although we note that $T_c(N)$ is ill-defined and $\gamma_N = \ln \alpha N/N$ leads to $a_c = 1$ for any fixed $\alpha > 0$) and with BEC for $a > 1$ and no BEC for $a \leqslant 1$. On the other hand, $T_c = \infty$ if $\gamma_N/(\ln N/N) \to \infty$ and $T_c = 0$ if $\gamma_N/(\ln N/N) \to 0$. In particular, for $\gamma_N = 1/N$ we find $T_c = 0$ and an extensive free energy and can, therefore, identify this scaling limit with that of the homogenous system. Since exact computations are possible, for the critical scaling we can obtain the asymptotic distribution of n_0, the number of particles in the ground state, and its mean value together with finite-size corrections. We find that for $a < 1$ the limit distribution of n_0/N^a is exponential and $\langle n_0 \rangle / N^a \to 1$. For $a > 1$ the condensate density $\langle n_0 \rangle / N \to 1 - a^{-1}$ and the fluctuation of n_0 is huge, $(\ln N/N)(n_0 - \langle n_0 \rangle)$ is asymptotically distributed following Gumbel's law. This holds true at the critical point $a = 1$ as well where $\langle n_0 \rangle = N \ln\ln N/\ln N + O(N/\ln N)$. Not surprisingly, the free energy is subextensive, $F_N^0 = -(\pi^2/6\beta a)N/\ln N + o(N/\ln N)$ for any $a > 0$, with no singularity at $a = 1$. A subtle difference compared with the three dimensional case, revealing itself only through the details described above, is that at all temperatures the gas forms a complete generalized Bose-Einstein condensate with the participation of at most $\sim N/\ln N$ one-particle levels. We prove this in Section 3.2. We also show there that for $a < 1$ a large number of low-lying levels are equally occupied by N^a particles, and for $a > 1$ particles not condensed into the one-particle ground state are in a generalized condensate at its critical point and there is no fragmented condensation, $\langle n_i \rangle = o(N)$ for each $i > 0$. In Section 3.3 we study the effect of interactions. We can only prove that at least a complete GBEC occurs at all temperatures provided that $\langle U_N \rangle_{\beta H_N^0}/N \ln N$ goes to zero as N tends to infinity. Here U_N is the N-particle interaction energy and $\langle U_N \rangle_{\beta H_N^0}$ is its mean value taken in the thermal equilibrium state of the noninteracting gas in the scaled trap. Thus $\langle U_N \rangle_{\beta H_N^0}$ can be of the order of N

or of slightly higher order, so that $F_N^0/\langle U_N\rangle_{\beta H_N^0} \to 0$, showing that the effect of the interaction is indeed nontrivial. For scaled pair interactions the analog of (4) is the somewhat weaker condition (stronger interaction) $b_N = o(\alpha_N \ln N/N)$. The mathematical analysis in Section 3 is conceptually not difficult but it is ramified and lengthy. Readers uninterested in details can go directly to the assertions of Theorems 3, 4 and 5. The paper ends with a Summary.

2. GENERALIZED BOSE CONDENSATION IN TRAPS

2.1. Preliminaries

The Hamiltonian of N noninteracting bosons is

$$H_N^0 = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m}\Delta_{\mathbf{x}_i} + V(\mathbf{x}_i) \right] = \oplus_{i=1}^{N} H^0. \tag{7}$$

We allow the confining potential V and thus the one-particle Hamiltonian H^0 to depend on N, and omit to indicate the resulting N-dependence of the eigenvalues $\varepsilon_0 < \varepsilon_1 \leqslant \varepsilon_2 \leqslant \cdots$ and eigenvectors φ_j. The N-particle interaction energy and the corresponding Hamiltonian will be denoted by U_N and $H_N = H_N^0 + U_N$, respectively. U_N is supposed to be bounded from below. As in ref. 14, for a positive integer J and a positive number δ we define the modified Hamiltonians $H^0(J,\delta)$, $H_N^0(J,\delta)$ and $H_N(J,\delta)$ by replacing ε_j with $\varepsilon_j + \delta$ for $j \leqslant J$ in the spectral resolution of H^0. For an operator A, $\langle A\rangle_{\beta H} = \mathrm{Tr}\, A\, e^{-\beta H} / \mathrm{Tr}\, e^{-\beta H}$ with the trace taken in the N-particle symmetric subspace. In particular, $\langle n_j\rangle_{\beta H}$ is the mean occupation number of φ_j.

The basic estimate we use for proving BEC or GBEC in the interacting gas is the following.

Lemma 2.1. For any $\beta > 0$

$$\begin{aligned}\sum_{j=0}^{J} \langle n_j\rangle_{\beta H_N} &\geqslant \sum_{j=0}^{J} \langle n_j\rangle_{\beta H_N^0(J,\delta)} - \frac{1}{\delta}[\langle U_N\rangle_{\beta H_N^0} - \inf U_N] \\ &\geqslant \langle n_0\rangle_{\beta H_N^0(J,\delta)} - \frac{1}{\delta}[\langle U_N\rangle_{\beta H_N^0} - \inf U_N]. \end{aligned} \tag{8}$$

This bound is a consequence of the identity

$$\sum_{j=0}^{J} \langle n_j \rangle_{\beta H_N^{(0)}(J,\delta)} = -\frac{\partial}{\partial(\beta\delta)} \ln \operatorname{Tr} e^{-\beta H_N^{(0)}(J,\delta)}, \tag{9}$$

the convexity of the logarithm of the trace as a function of $\beta\delta$ and Bogoliubov's convexity inequality[26]

$$\ln \frac{\operatorname{Tr} e^{-\beta H_1}}{\operatorname{Tr} e^{-\beta H_2}} \geqslant -\beta \langle H_1 - H_2 \rangle_{\beta H_2}, \tag{10}$$

cf. the first part of the proof of Theorem 4.2 in ref. 14.

Lemma 2.1 is useless in the case of hard-core interactions which yield $-\infty$ on the right-hand side of the inequality (8), but in the case of integrable interactions it leads to nontrivial results. Suppose that V is fixed, and

$$U_N(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sum_{i<j} u_N(\mathbf{x}_i - \mathbf{x}_j) \tag{11}$$

One can show that

$$\frac{1}{N} |\langle U_N \rangle_{\beta H_N^0} - (\Phi_0, U_N \Phi_0)| \leqslant c(\beta) \|u_N\|_1 \tag{12}$$

where $\Phi_0 = \varphi_0(\mathbf{x}_1) \cdots \varphi_0(\mathbf{x}_N)$, $\|u_N\|_1 = \int |u_N(\mathbf{x})| \, d\mathbf{x}$ and $c(\beta)$ is independent of N. The above lemma and the bound (12) were used earlier to prove a theorem (Theorem 4.2 of ref. 14) that we repeat here in a slightly different form.

Theorem 1. Suppose that V is independent of N. Let U_N be a positive pair interaction with $\|u_N\|_1 \leqslant C/N$ for some constant C. Then

$$L(U) \equiv \lim_{N\to\infty} (\Phi_0, U_N \Phi_0)/N < \infty, \tag{13}$$

and for any $\beta > 0$ and $J \geqslant 0$

$$\sum_{j=0}^{J} \lim_{N\to\infty} \frac{1}{N} \langle n_j \rangle_{\beta H_N} \geqslant 1 - \frac{L(U)}{\varepsilon_{J+1} - \varepsilon_0}. \tag{14}$$

The main example is (2) with (4). Because $L(U)$ is finite and ε_j are independent of N and tend to infinity, there exists a J_0 independent of N and uniquely defined through

$$\varepsilon_{J_0} - \varepsilon_0 \leqslant L(U) < \varepsilon_{J_0+1} - \varepsilon_0. \tag{15}$$

The right-hand side of (14) is positive for $J \geqslant J_0$, implying that at least one of $\varphi_0 \ldots, \varphi_{J_0}$ is macroscopically occupied. This is just Bose-Einstein condensation, because of the following simple result, shown in ref. 14.

Lemma 2.2. Let $\sigma(N)$ be any N-particle density matrix and φ any normalized element of the one-particle Hilbert space $\mathcal{H}$. Then

$$\langle n[\varphi]\rangle_\sigma = (\varphi, \sigma_1 \varphi) \tag{16}$$

where $\sigma_1(N)$ is the one-particle reduced density matrix corresponding to σ and $n[\varphi]$ is the occupation number operator in $\mathcal{H}^N$, associated to φ,

$$n[\varphi] = |\varphi\rangle\langle\varphi| \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes |\varphi\rangle\langle\varphi|. \tag{17}$$

Thus $(\varphi_j, \sigma_1 \varphi_j) = \langle n[\varphi_j]\rangle_{\beta H_N} \equiv \langle n_j\rangle_{\beta H_N}$ is of the order of N for at least one of $j = 0, \ldots, J_0$, and the maximum eigenvalue of σ_1 satisfies

$$\|\sigma_1\| \geqslant \frac{N}{J_0+1}\left(1 - \frac{L(U)}{\varepsilon_{J_0+1} - \varepsilon_0}\right), \tag{18}$$

implying BEC. Although in ref. 14, the theorem was proved for a general J as presented above, it was stated only for $J = J_0$, because we were interested in finding the minimum number of eigenstates of H^0 one of which can be seen to be occupied macroscopically. In fact, Theorem 1 implies a *complete* BEC at all temperatures. To see this, it suffices to take the limit $J \to \infty$ in (14), yielding

$$\sum_{j=0}^{\infty} \lim_{N\to\infty} \frac{1}{N} \langle n_j\rangle_{\beta H_N} = 1. \tag{19}$$

Only $\langle n_j\rangle_{\beta H_N} \propto N$ contribute to the sum. Thus, all but an asymptotically vanishing fraction of particles are carried by macroscopically occupied one-particle eigenstates. It is not difficult to show that the same is true for the eigenstates of σ_1. If $\sigma_1\psi_j = \lambda_j\psi_j$, $j=0,1\ldots$,

$$\sum_{j=0}^{\infty} \lim_{N\to\infty} \frac{1}{N}\langle n[\psi_j]\rangle_{\beta H_N} = \sum_{j=0}^{\infty} \lim_{N\to\infty} \lambda_j(N)/N = 1 \tag{20}$$

comes from (14) and the following generalization of the variational principle.

Lemma 2.3. Let A be an upper semibounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Suppose that A has a pure point spectrum $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$. Then for any positive integer n

$$\sum_{i=1}^{n} \lambda_i = \sup_{\substack{\phi_1,\ldots,\phi_n \in \mathcal{H} \\ (\phi_i,\phi_j)=\delta_{ij}}} \sum_{i=1}^{n} (\phi_i, A\phi_i). \tag{21}$$

Proof. Let ψ_i be the orthonormal eigenvectors of A. Because $\lambda_i = (\psi_i, A\psi_i)$, the left side of (21) is smaller than or equal to the right side. Therefore, only the opposite inequality is to be shown. Choose any orthonormal set $\phi_1,\ldots,\phi_n$ of vectors of $\mathcal{H}$ and let $\phi_i = \sum_{j=1}^{\infty} a_{ij}\psi_j$. Then

$$\begin{aligned}
\sum_{i=1}^{n}(\phi_i, A\phi_i) &= \sum_{i=1}^{n}\sum_{j=1}^{\infty}|a_{ij}|^2\lambda_j = \sum_{i=1}^{n}\left[\sum_{j=1}^{n}|a_{ij}|^2\lambda_j + \sum_{j=n+1}^{\infty}|a_{ij}|^2\lambda_j\right] \\
&\leqslant \sum_{i=1}^{n}\left[\sum_{j=1}^{n}|a_{ij}|^2\lambda_j + \lambda_{n+1}\left(1-\sum_{j=1}^{n}|a_{ij}|^2\right)\right] \\
&= n\lambda_{n+1} + \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2(\lambda_j - \lambda_{n+1}) \\
&= n\lambda_{n+1} + \sum_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}|^2\right)(\lambda_j - \lambda_{n+1}) \\
&\leqslant n\lambda_{n+1} + \sum_{j=1}^{n}(\lambda_j - \lambda_{n+1}) = \sum_{j=1}^{n}\lambda_j,
\end{aligned} \tag{22}$$

which remains valid if we take the supremum in the leftmost member. ∎

Applying this lemma to $A=\sigma_1(N)$ we see that $\sum_{j=0}^{J}\langle n_j\rangle_{\beta H_N}\leqslant\sum_{j=0}^{J}\lambda_j$ for any J. From (14), therefore,

$$\sum_{j=0}^{J}\lim_{N\to\infty}\lambda_j(N)/N\geqslant 1-\frac{L(U)}{\varepsilon_{J+1}-\varepsilon_0}. \tag{23}$$

Lettig J tend to infinity we obtain (20). For Gross-Pitaevskii scaling in three dimensions Seiringer[17] proved the much stronger result $\lim_{N\to\infty}\lambda_0(N)/N=1$.

The logic of the use of Lemma 2.1 for proving GBEC is somewhat different from the strategy followed in Theorem 1: First we make an appropriate choice of J and then we impose the necessary condition on the interaction. An obvious consequence of Lemma 2.1 is

Proposition 2.1. Let $J=J(N)$ with $J=o(N)$ or $J=\lfloor sN\rfloor$. Choose $\delta=\varepsilon_{J+1}-\frac{1}{2}(\varepsilon_0+\varepsilon_1)$. Fix $\beta>0$ and suppose that

$$\lim\frac{1}{N}\left[\sum_{j=0}^{J}\langle n_j\rangle_{\beta H_N^0(J,\delta)}-\frac{1}{\delta}[\langle U_N\rangle_{\beta H_N^0}-\inf U_N]\right]=b>0 \tag{24}$$

where lim means $\lim_{N\to\infty}$ if $J/N\to 0$ and $\lim_{s\to 0}\lim_{N\to\infty}$ if $J=\lfloor sN\rfloor$. Then

$$\lim\frac{1}{N}\sum_{j=0}^{J}\langle n_j\rangle_{\beta H_N}\geqslant b. \tag{25}$$

The inequality (25) implies *at least* GBEC. Note that $\delta=\delta(J,N)$ with a separate dependence on N through the eigenvalues of H^0 if V is scaled. In some applications the positivity of the left member of (24) can hold with a single term, $j=0$, of the sum.

The results of the theorem and the proposition above apply to the ground state as well, if we take first the limit $\beta\to\infty$, then $N\to\infty$. In the use of Lemma 2.1 for proving BEC or GBEC the lower part of the spectrum of H^0 is shifted upwards in such a way that the ground state of the modified Hamiltonian $H^0(J,\delta)$ remains φ_0, the ground state of H^0. Therefore

$$\lim_{\beta\to\infty}\langle n_0\rangle_{\beta H_N^0(J,\delta)}=N \tag{26}$$

and there is BEC or GBEC in the ground state of the interacting gas provided that

$$\lim \frac{1}{N\delta}[(\Phi_0, U_N\Phi_0) - \inf U_N] < 1. \tag{27}$$

It is interesting to see why and how the method described above fails in proving BEC or GBEC of the homogenous Bose gas. We exhibit this failure on the ground state. Let us consider first the one-dimensional homogenous δ-gas, that is, $u(x) = 2c\delta(x)$ and

$$V(x) = \begin{cases} 0 & 0 < x < L \\ +\infty & x \leqslant 0, \quad x \geqslant L \end{cases} \tag{28}$$

or take a periodic boundary condition at 0 and L. Let $\rho = N/L$. In the periodic case $(\Phi_0, U_N\Phi_0) = cN(N-1)/L$ and therefore GBEC follows from Proposition 2.1 and equation (27) if $\lim J/N = 0$ and $c\rho < \delta \approx \varepsilon_J \propto J^2/L^2 = \rho^2(J/N)^2$ which means $\lim c/\rho = \lim (J/N)^2 = 0$, i.e., a vanishing interaction. Still, Proposition 2.1 yields a nontrivial estimate for interacting homogenous gases. Let, in general, $U_N = \sum_{i<j} u(\mathbf{x}_i - \mathbf{x}_j) \geqslant -BN$, $\int |u| \, d\mathbf{x} < \infty$, $\rho > 0$ and consider N bosons in a d-dimensional cube of side $L = (N/\rho)^{1/d}$. For periodic boundary conditions $(\Phi_0, U_N\Phi_0) = \frac{1}{2}\rho(N-1)\int u \, d\mathbf{x}$ and therefore (27) holds true if $\delta > \frac{1}{2}\rho \int u \, d\mathbf{x} + B > 0$. Recalling that in the setup of the proposition $\lim_{N\to\infty} \varepsilon_J/\delta = 1$ if $J \to \infty$ with N, choose

$$J(N) = [v_d (2m\delta)^{d/2}/\rho h^d] N \tag{29}$$

where v_d is the volume of the unit ball in d dimensions. Then the zero temperature version of Proposition 2.1 for homogenous gases proves

$$\lim_{N\to\infty} \lim_{\beta\to\infty} \frac{1}{N} \sum_{j=0}^{J(N)} \langle n_j \rangle_{\beta H_N} \geqslant 1 - \frac{\frac{1}{2}\rho \int u \, d\mathbf{x} + B}{\delta}, \tag{30}$$

i.e., a macroscopic number of particles are distributed over a macroscopic number (29) of lowest lying levels. Although this does not prove BEC, the bound (30) is nontrivial because the number of levels is infinite for finite N.

To use Proposition 2.1 at positive temperatures, BEC or GBEC has to be proven in the noninteracting gas with the shifted spectrum. This is a minor problem if the external potential is fixed, as in the case of our forthcoming discussion of GBEC in superharmonic traps. For the scaled harmonic potential we will have to pay somewhat more attention to this question.

2.2. Superharmonic Traps in One Dimension

In one dimension the particularity of potentials with $x^2/V(x)\to 0$ as $|x|\to\infty$ is that $n/\varepsilon_n\to 0$ as $n\to\infty$. As an example, for homogenous potentials $V(x)=c|x|^\eta$ semiclassical quantization yields the eigenvalues in the form

$$\varepsilon_n=\left[\frac{h\eta c^{1/\eta}}{4\sqrt{2m}I_\eta}(n+O(1))\right]^{\frac{2\eta}{2+\eta}}$$
$$I_\eta=\int_0^1 x^{-1+1/\eta}\sqrt{1-x}\,\mathrm{d}x=\frac{\Gamma(\frac{3}{2})\Gamma(1/\eta)}{\Gamma(\frac{3}{2}+1/\eta)},\qquad(31)$$

showing that the eigenvalues increase faster than linearly if $\eta>2$. Note that Sturmian theory[27] confirms the semiclassical formula (31).

Let, therefore, H^0 be a fixed one-particle Hamiltonian with a spectrum such that $n/\varepsilon_n\to 0$ (which is our definition of a superharmonic trap in one dimension) and choose $J(N)\to\infty$ in such a way that $\lim J/N=0$ and $\lim N/\varepsilon_J=0$. Then for $\delta=\varepsilon_{J+1}-\frac{1}{2}(\varepsilon_0+\varepsilon_1)$ we also have $\lim N/\delta=0$.

In ref. 14 we proved that in the case of a noninteracting gas in a fixed trap there exists a uniform upper bound on the mean value of $N'=N-n_0$, namely, for any N and any $\mu<\varepsilon_1-\varepsilon_0$

$$\langle N'\rangle_{\beta H_N^0}\leqslant\frac{1}{(1-e^{-\beta\mu})^2}\prod_{n=1}^{\infty}\frac{1}{1-e^{-\beta(\varepsilon_n-\varepsilon_0-\mu)}}.\qquad(32)$$

Similar inequality holds for $\langle N'\rangle_{\beta H_N^0(J,\delta)}$, with the exception that the bound still depends on N because $\varepsilon_n-\varepsilon_0$ has to be replaced by $\varepsilon_n-\varepsilon_0-\delta$ if $n>J$. However, due to superharmonicity, for $n\geqslant J+2$ we can use the estimate

$$\varepsilon_n-\varepsilon_0-\delta=\varepsilon_n-\varepsilon_{J+1}+\frac{1}{2}(\varepsilon_1-\varepsilon_0)>\varepsilon_{n-J-1}-\varepsilon_0\qquad(33)$$

if N (and thus J) is large enough. Choosing e.g., $\mu=\frac{1}{4}(\varepsilon_1-\varepsilon_0)$, for sufficiently large N we obtain

$$\langle N'\rangle_{\beta H_N^0(J,\delta)} \leqslant \frac{1}{(1-e^{-\beta\mu})^3}\left[\prod_{n=1}^{\infty}\frac{1}{1-e^{-\beta(\varepsilon_n-\varepsilon_0-\mu)}}\right]^2. \tag{34}$$

Therefore

$$\lim \frac{1}{N}\langle n_0\rangle_{\beta H_N^0(J,\delta)} = 1 \tag{35}$$

and the inequality (24) holds if

$$\lim \frac{1}{N\delta}[\langle U_N\rangle_{\beta H_N^0} - \inf U_N] < 1. \tag{36}$$

In words, GBEC follows if the mean interaction energy $\langle U_N\rangle_{\beta H_N^0}$ does not exceed $N\delta \approx N\varepsilon_J$, the maximum energy of N noninteracting particles in the would-be (generalized) condensate.

If U_N is a pair interaction, we can use the estimate (12) and

$$|(\Phi_0, U_N\Phi_0)| \leqslant \frac{N(N-1)}{2}\|u_N\|_1\|\varphi_0^4\|_1 \tag{37}$$

to obtain

$$\begin{aligned}&\lim \frac{1}{N\delta}[\langle U_N\rangle_{\beta H_N^0} - \inf U_N]\\ &\quad\leqslant \lim \frac{1}{\delta}\left[\left(\frac{N-1}{2}\|\varphi_0^4\|_1 + c(\beta)\right)\|u_N\|_1 - \frac{\inf U_N}{N}\right].\end{aligned} \tag{38}$$

If the interaction is stable, $\inf U_N/N \geqslant -B$ for some constant B; if it is not stable still $\inf U_N/N \geqslant -(N-1)\inf u_N/2$. In either case, for $u_N = u$ integrable, bounded from below and independent of N the quantity in the square bracket is of the order of N and thus the limit in (38) vanishes. With Lemma 2.3 this implies

Theorem 2. In one-dimensional superharmonic traps bosons interacting via a lower semibounded (unscaled) integrable pair interaction undergo a complete generalized Bose-Einstein condensation at all temperatures: For any choice of $J(N)$ such that $\lim J/N = 0$ and $\lim N/\varepsilon_J = 0$

$$\lim \frac{1}{N}\sum_{j=0}^{J}\lambda_j(N) = \lim \frac{1}{N}\sum_{j=0}^{J}\langle n_j\rangle_{\beta H_N} = 1. \tag{39}$$

The claim of the theorem could be made stronger. For instance, in a box (28) with a fixed L independent of N, J and $\delta \sim \varepsilon_J$ can grow almost as fast as N and N^2, respectively, and therefore we can obtain GBEC for interactions as strong as $\langle U_N \rangle = o(N^3)$ instead of $O(N^2)$ stated in the theorem.

3. SCALED HARMONIC TRAP IN ONE DIMENSION

3.1. Bose Condensation via Phase Transition in the Noninteracting Gas

The one-particle Hamiltonian is

$$H^0 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2 x^2. \tag{40}$$

Measuring the energy from that of the ground state, the partition function for N noninteracting particles reads

$$Q_{N,a} = \sum_{\{n_j\}_{j\geqslant 0}:\sum n_j = N} e^{-a\sum jn_j} = \sum_{\{n_j\}_{j>0}:\sum n_j \leqslant N} e^{-a\sum jn_j} = \sum_{m=0}^{N} q_{m,a} \tag{41}$$

with

$$q_{m,a} = \sum_{\{n_j\}_{j>0}:\sum n_j = m} e^{-a\sum jn_j} \tag{42}$$

where we have introduced the notation $a = \hbar\omega\beta$ and used $\beta(\varepsilon_j - \varepsilon_0) = ja$. The key to the forthcoming analysis is

Lemma 3.1.

$$Q_{N,a} = \prod_{k=1}^{N}(1 - e^{-ka})^{-1} \tag{43}$$

$$q_{m,a} = e^{-ma}\prod_{k=1}^{m}(1 - e^{-ka})^{-1} \tag{44}$$

and the probability of having m particles in excited states is

$$P_{N,a}(N'=m)=e^{-ma}\prod_{k=m+1}^{N}(1-e^{-ka}). \tag{45}$$

Proof. $P_{N,a}(N'=m)=q_{m,a}/Q_{N,a}$ and (44) follows from (43) through $q_{m,a}=Q_{m,a}-Q_{m-1,a}$. Now $Q_{N,a}$ can be rewritten as

$$Q_{N,a}=\sum_{0\leqslant i_1\leqslant\cdots\leqslant i_N} e^{-a\sum_{j=1}^{N} i_j}=\sum_{i_1=0}^{\infty}e^{-i_1 a}\sum_{i_2=i_1}^{\infty}e^{-i_2 a}\cdots\sum_{i_N=i_{N-1}}^{\infty}e^{-i_N a} \tag{46}$$

from which (43) follows.

An alternative way is to compute first $q_{m,a}$ by using that it is the generating function of $p_m(n)$, the number of (unordered) partitions of n into m parts,

$$q_{m,a}=\sum_{n=m}^{\infty}p_m(n)e^{-na}=\sum_{n=1}^{\infty}p_m(n)e^{-na} \tag{47}$$

because $p_m(n)=0$ for $n<m$. Starting with the identity

$$p_m(n)=p_m(n-m)+p_{m-1}(n-m)+\cdots+p_1(n-m), \tag{48}$$

valid for $n>m\geqslant 1$, one can derive the recurrence relation

$$q_{m,a}=\frac{e^{-ma}}{1-e^{-ma}}\sum_{k=0}^{m-1}q_{k,a} \qquad q_{0,a}\equiv 1 \tag{49}$$

from which (44) follows by induction and (43) by (41) and (49). ∎

Due to the simple form of $P_{N,a}(N'=m)$ we can obtain precise asymptotic results on the distribution of n_0 in the case of different scalings. In what follows, we discuss the thermodynamics of the noninteracting gas in a scaled harmonic trap, characterized by the Hamiltonian

$$H_N^0=\sum_{i=1}^{N}\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x_i^2}+\gamma_N V(\sqrt{\gamma_N}x_i)\right] \tag{50}$$

where $V(x) = \frac{1}{2}m\omega^2 x^2$ and $\gamma_N \to 0$. This amounts to open the trap by replacing ω by $\omega\gamma_N$ or, equivalently, $a = \hbar\omega\beta$ by

$$a_N = a\gamma_N \to 0. \tag{51}$$

From Lemma 3.1

$$P_{N,a_N}(N' \leqslant m) = \frac{Q_{m,a_N}}{Q_{N,a_N}} = \prod_{l=m+1}^{N} (1 - e^{-la_N}). \tag{52}$$

Let $\lambda_N = 1 - m/N$ where m is any integer between 0 and N. Taking the logarithm of equation (52) and expanding the right member in Taylor series we find

$$\ln P_{N,a_N}\left(\frac{n_0}{N} \geqslant \lambda_N\right) = -\sum_{k=1}^{\infty} A_{Nk},$$
$$A_{Nk}(\lambda_N) = \frac{1}{k}\sum_{l=m+1}^{N} e^{-kla_N} = \frac{e^{-kNa_N}(e^{k\lambda_N N a_N} - 1)}{k(e^{ka_N} - 1)}. \tag{53}$$

$A_{Nk}(\lambda_N)$ are nonnegative, monotone decreasing with k and increasing with λ_N. Later on,

$$P_{N,a_N}\left(\frac{n_0}{N} \geqslant \lambda_N\right) \leqslant e^{-A_{N1}(\lambda_N)} \tag{54}$$

will be used.

The following two propositions serve to find necessary conditions on the sequence a_N which give rise to a trivial asymptotic distribution of n_0/N. In this section we use the notation $\langle n_0\rangle_{N,a_N}$ for $\langle n_0\rangle_{\beta H_N^0}$.

Proposition 3.1. Suppose that

$$\lambda_N \to \lambda > 0 \quad \text{and} \quad \ln N - \left(1 - \frac{\lambda_N}{2}\right) N a_N \to \infty. \tag{55}$$

Then

$$P_{N,a_N}\left(\frac{n_0}{N} \geqslant \lambda_N\right) \to 0 \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N}\langle n_0\rangle_{N,a_N} \leqslant \lambda. \tag{56}$$

If $\ln N - Na_N \to \infty$ then $\frac{1}{N}\langle n_0\rangle_{N,a_N} \to 0$, i.e., there is no Bose-Einstein condensation.

Proof. To obtain the vanishing probability it suffices to show that $A_{N1}\to\infty$. From the inequalities

$$xe^{x/2}<e^x-1<xe^x \tag{57}$$

valid for $x>0$,

$$A_{N1}>e^{\ln N-(1-\lambda_N/2)Na_N+\ln\lambda_N-a_N}\to\infty \tag{58}$$

indeed. Moreover,

$$\frac{1}{N}\langle n_0\rangle_{N,a_N}\leqslant\lambda_N P_{N,a_N}\left(\frac{n_0}{N}<\lambda_N\right)+P_{N,a_N}\left(\frac{n_0}{N}\geqslant\lambda_N\right)\to\lambda. \tag{59}$$

If the stronger condition $\ln N-Na_N\to\infty$ is fulfilled, (55) and therefore (56) hold true for all $\lambda>0$. This implies the absence of BEC. ∎

The counterpart of Proposition 3.1 is

Proposition 3.2. Suppose that

$$\lambda_N\to\lambda\leqslant 1\quad\text{and}\quad(1-\lambda_N)Na_N-\ln N\to\infty. \tag{60}$$

Then

$$P_{N,a_N}\left(\frac{n_0}{N}\geqslant\lambda_N\right)\to 1\quad\text{and}\quad\lim_{N\to\infty}\frac{1}{N}\langle n_0\rangle_{N,a_N}\geqslant\lambda. \tag{61}$$

If $Na_N/\ln N\to\infty$ then $\frac{1}{N}\langle n_0\rangle_{N,a_N}\to 1$, i.e., there is a complete Bose-Einstein condensation.

Proof. Using (57) and

$$-\ln(1-e^{-x})<\frac{1}{e^x-1}\qquad(x>0)$$

we obtain

$$0<\sum_{k=1}^{\infty}A_{Nk}<N\lambda_N\left|\ln\left(1-e^{-(1-\lambda_N)Na_N}\right)\right|<\frac{\lambda_N}{e^{(1-\lambda_N)Na_N-\ln N}-\frac{1}{N}}\to 0 \tag{62}$$

which implies the result on the limit of the probability. On the other hand,

$$\frac{1}{N}\langle n_0\rangle_{N,a_N} \geqslant \lambda_N P_{N,a_N}\left(\frac{n_0}{N} \geqslant \lambda_N\right) \to \lambda. \tag{63}$$

Suppose now that $Na_N/\ln N \to \infty$. One can choose a sequence $\lambda_N \to 1$ in such a way that $(1-\lambda_N)Na_N/\ln N \to \infty$. Then (60) is fulfilled and (63) holds for $\lambda = 1$. ▮

From Propositions 3.1 and 3.2 we can identify the scalings which yield trivial limits:

Corollary 3.1. If $N\gamma_N/\ln N \to 0$ then $T_c = 0$. If $N\gamma_N/\ln N \to \infty$ then $T_c = \infty$ and there is a complete BEC at all $T < \infty$.

Proof. In the first case $Na_N/\ln N \to 0$ and thus $\ln N - Na_N \to \infty$ for any $a < \infty$ i.e., $T > 0$, and Proposition 3.1 applies. In the second case $Na_N/\ln N \to \infty$ for any $a > 0$ i.e., $T < \infty$, and Proposition 3.2 applies. ▮

Thus, we have found that a phase transition can occur only if $N\gamma_N/\ln N$ has a finite nonvanishing limit. In this case without restricting generality we suppose that

$$\gamma_N = \ln N/N, \quad a_N = a\ln N/N. \tag{64}$$

Choosing a different prefactor can only change the critical temperature. Propositions 3.1 and 3.2 already yield the phase transition:

Corollary 3.2. If $a_N = a\ln N/N$ then $a_c = 1$. For $a < 1$ there is no BEC. For $a > 1$ and $0 < \lim \lambda_N < 1 - a^{-1}$

$$\lim_{N\to\infty} P_{N,a_N}\left(\frac{n_0}{N} \geqslant \lambda_N\right) = 1. \tag{65}$$

Moreover,

$$\lim_{N\to\infty} \frac{1}{N}\langle n_0\rangle_{N,a_N} \geqslant 1 - \frac{1}{a}. \tag{66}$$

Proof. If $a < 1$, $\ln N - Na_N = (1-a)\ln N \to \infty$ and by Proposition 3.1 there is no BEC. If $a > 1$ and $0 < \lambda = \lim \lambda_N < 1 - a^{-1}$ then (60) and thus (61) hold true. Letting λ tend to $1 - a^{-1}$ we find (66). ▮

This is a temporary result. We will show, among others, that for $a \geqslant 1$ the probability of having $n_0/N > 1-a^{-1}$ tends to zero and thus the distribution of n_0/N becomes degenerate and concentrated on $1-a^{-1}$. This will imply that in (66) there is equality.

Proposition 3.3. Suppose that

$$\lambda_N \to \lambda \leqslant 1 \quad \text{and} \quad (1-\lambda_N)Na_N \to \infty. \tag{67}$$

Then

$$P_{N,a_N}\left(\frac{n_0}{N} \geqslant \lambda_N\right) = e^{-(1+\epsilon_N)A_{N1}(\lambda_N)} \tag{68}$$

where $0 < \epsilon_N < -\ln\left(1-e^{-(1-\lambda_N)Na_N}\right) \to 0$.

Proof. Let us rewrite equation (53) in the form

$$\ln P_{N,a_N}\left(\frac{n_0}{N} \geqslant \lambda_N\right) = -A_{N1}\left(1+\sum_{k=1}^{\infty}\frac{A_{N,k+1}}{A_{N1}}\right). \tag{69}$$

Now

$$\begin{aligned}\frac{A_{N,k+1}}{A_{N1}} &= \frac{e^{-(m+1)(k+1)a_N}}{e^{-(m+1)a_N}}\frac{1}{k+1}\frac{\sum_{l=1}^{N-m-1} e^{-l(k+1)a_N}}{\sum_{l=1}^{N-m-1} e^{-la_N}} \\ &< \frac{1}{k+1}e^{-(m+1)ka_N} < \frac{1}{k}e^{-k(1-\lambda_N)Na_N}\end{aligned} \tag{70}$$

and therefore

$$\sum_{k=1}^{\infty}\frac{A_{N,k+1}}{A_{N1}} < \sum_{k=1}^{\infty}\frac{1}{k}e^{-k(1-\lambda_N)Na_N} = -\ln\left(1-e^{-(1-\lambda_N)Na_N}\right) \to 0 \tag{71}$$

as $N \to \infty$. ▮

Henceforth, we concentrate on the phase transition. With the scaling (64) the condition (67) is satisfied if $a > 0$ and $\lambda < 1$. Most of the results listed in the theorem below follow from Proposition 3.3 applied to this case,

$$P_{N,a_N}\left(\frac{n_0}{N} \geqslant \lambda_N\right) = \exp\left\{-N^{-a}\frac{N^{a\lambda_N}-1}{e^{a\ln N/N}-1}(1+\epsilon_N)\right\}. \tag{72}$$

Theorem 3. The scaling $a_N = a \ln N/N$ leads to a phase transition at $a \equiv \hbar\omega\beta = 1$ with no BEC for $a \leqslant 1$ and BEC for $a > 1$. In details, the following hold true.

I. Limit distribution of n_0.

(i) For $0 < a < 1$ and $x \geqslant 0$

$$\lim_{N\to\infty} P_{N,a_N}\left(\frac{n_0}{N^a} \geqslant x\right) = e^{-x}. \tag{73}$$

(ii) For $a \geqslant 1$ and $\lambda_N \to \lambda \in]0, 1[$

$$\lim_{N\to\infty} P_{N,a_N}\left(\frac{n_0}{N} \geqslant \lambda_N\right) = \begin{cases} 1 \text{ if } \lambda < 1 - a^{-1} \\ 0 \text{ if } \lambda > 1 - a^{-1}. \end{cases} \tag{74}$$

(iii) For $a \geqslant 1$ and any real x

$$\lim_{N\to\infty} P_{N,a_N}\left(\left[\frac{n_0}{N} - 1 + a^{-1}\right]\ln N - a^{-1}\ln\ln N \geqslant x\right) = \exp\left\{-a^{-1}e^{ax}\right\}. \tag{75}$$

Equivalently,

$$\lim_{N\to\infty} P_{N,a_N}\left(\frac{\ln N}{N}(n_0 - \langle n_0\rangle_{N,a_N}) \geqslant x\right) = \exp\left\{-a^{-1}e^{a(x+\eta(a))}\right\} \tag{76}$$

where

$$\eta(a) = \int_{-\infty}^{\infty} x \exp\left\{ax - a^{-1}e^{ax}\right\} \mathrm{d}x. \tag{77}$$

II. Mean value of n_0.

(i) For $0 < a < 1$

$$\lim_{N\to\infty} \frac{\langle n_0\rangle_{N,a_N}}{N^a} = 1. \tag{78}$$

(ii) For $a \geqslant 1$

$$\langle n_0\rangle_{N,a_N} = (1 - a^{-1})N + \frac{N}{\ln N}[a^{-1}\ln\ln N + \eta(a)] + o\left(\frac{N}{\ln N}\right). \tag{79}$$

In particular,

$$\lim_{N\to\infty}\frac{1}{N}\langle n_0\rangle_{N,a_N}=1-\frac{1}{a}. \tag{80}$$

III. For any $a>0$ the free energy of N particles is

$$F_N^0=-\frac{\pi^2}{6\beta a}\frac{N}{\ln N}+o\left(\frac{N}{\ln N}\right). \tag{81}$$

Proof. (I.i) Substitute $\lambda_N=x/N^{1-a}$ into Eq. (72). Because $\lambda_N\ln N\to 0$, $N^{a\lambda_N}-1$ can be replaced by $a\lambda_N\ln N$ and (73) follows.
(I.ii) If $\lambda<1-a^{-1}$ then $A_{N1}(\lambda_N)\to 0$ and if $\lambda>1-a^{-1}$ then $A_{N1}(\lambda_N)\to\infty$. Therefore the limit of Eq. (72) is the degenerate distribution (74).
(I.iii) The Gumbel distribution (75) is obtained by substituting

$$\lambda_N=1-\frac{1}{a}+\frac{\ln\ln N+ax}{a\ln N} \tag{82}$$

into Eq. (72). Because $\lambda_N\to 1-a^{-1}<1$, condition (67) is satisfied and Proposition 3.3 indeed applies. The form Eq. (76) follows from (II.ii) to be shown below; $\eta(a)$ is the expectation value of the Gumbel distribution.
(II.i) For any $\Delta>0$

$$\begin{aligned}&\sum_{m=1}^{\infty}m\Delta\left[P_{N,a_N}\left(\frac{n_0}{N^a}\geqslant m\Delta\right)-P_{N,a_N}\left(\frac{n_0}{N^a}\geqslant(m+1)\Delta\right)\right]\\&\quad\leqslant\frac{\langle n_0\rangle_{N,a_N}}{N^a}\\&\quad\leqslant\sum_{m=1}^{\infty}m\Delta\left[P_{N,a_N}\left(\frac{n_0}{N^a}\geqslant(m-1)\Delta\right)-P_{N,a_N}\left(\frac{n_0}{N^a}\geqslant m\Delta\right)\right].\end{aligned} \tag{83}$$

The sums are actually finite but the upper bounds tend to infinity with N. Using the inequalities (54) and (57)

$$P_{N,a_N}\left(\frac{n_0}{N^a}\geqslant x\right)<\exp\{-A_{N1}(xN^{-1+a})\}<\exp\{-xe^{-a\ln N/N}\}<e^{-x/2} \tag{84}$$

for N large enough. For the left and right members of Eq. (83) $m\Delta e^{-(m-1)\Delta/2}$ is a summable upper bound, thus we can interchange the limit $N\to\infty$

and the summation over m to find, with Eq. (73) and the convexity of the exponential,

$$e^{-\Delta}\sum_{m=1}^{\infty} m\Delta e^{-m\Delta}\Delta \leqslant \sum_{m=1}^{\infty} m\Delta(e^{-m\Delta}-e^{-(m+1)\Delta}) \leqslant \lim_{N\to\infty}\frac{\langle n_0\rangle_{N,a_N}}{N^a} \leqslant \sum_{m=1}^{\infty} m\Delta(e^{-(m-1)\Delta}-e^{-m\Delta}) \leqslant e^{\Delta}\sum_{m=1}^{\infty} m\Delta e^{-m\Delta}\Delta. \tag{85}$$

Letting Δ tend to zero the Riemann sums go to $\int_0^\infty xe^{-x}\,\mathrm{d}x=1$.
(II.ii) We start as before. Let

$$f_N(n_0)=\left(\frac{n_0}{N}-1+a^{-1}\right)\ln N - a^{-1}\ln\ln N. \tag{86}$$

For any $\Delta>0$

$$\begin{aligned}&\sum_{m=-\infty}^{\infty} m\Delta\left[P_{N,a_N}\left(f_N(n_0)\geqslant m\Delta\right)-P_{N,a_N}\left(f_N(n_0)\geqslant(m+1)\Delta\right)\right]\\ &\quad\leqslant\langle f_N(n_0)\rangle_{N,a_N}\\ &\quad\leqslant\sum_{m=-\infty}^{\infty} m\Delta\left[P_{N,a_N}\left(f_N(n_0)\geqslant(m-1)\Delta\right)-P_{N,a_N}\left(f_N(n_0)\geqslant m\Delta\right)\right].\end{aligned} \tag{87}$$

The sums are finite with the upper and lower bounds tending to infinity as N increases. Suppose that we can interchange the summation with the limit $N\to\infty$. Then Eq. (75) yields

$$\begin{aligned}&\sum_{m=-\infty}^{\infty} m\Delta\left[e^{-a^{-1}e^{am\Delta}}-e^{-a^{-1}e^{a(m+1)\Delta}}\right]\\ &\quad\leqslant\lim_{N\to\infty}\langle f_N(n_0)\rangle_{N,a_N}\leqslant\sum_{m=-\infty}^{\infty} m\Delta\left[e^{-a^{-1}e^{a(m-1)\Delta}}-e^{-a^{-1}e^{am\Delta}}\right].\end{aligned} \tag{88}$$

Now $\exp\{-a^{-1}e^{ax}\}$ is concave if $x<a^{-1}\ln a$ and convex if $x>a^{-1}\ln a$. Accordingly, we divide the sums in two parts, bound the differences in the square brackets with Δ times the derivatives at the upper or lower end of the intervals and let Δ go to zero. Both the upper and lower bound of $\langle f_N(n_0)\rangle_{N,a_N}$ converge to $\eta(a)$. Equation (79) follows by simple rearrangement. In Eq. (87) the interchange of the summation with the limit $N\to\infty$ is again based on the dominated convergence theorem. However, the sums have to be divided in two parts. For $m>0$ we can use Eq. (54) with Eq. (82) and $x=(m-1)\Delta$ as an upper bound on the difference (actually on both terms) in the square brackets because $A_{N1}(\lambda_N)\geqslant(2a)^{-1}e^{ax}$ if N is large enough. For $m<0$ only the difference in the square bracket is small. Using Eq. (71) a lengthy but straightforward computation yields

$$\begin{aligned}P_{N,a_N}(f_N(n_0)\ &\geqslant\ m\Delta)-P_{N,a_N}\,(f_N(n_0)\geqslant(m+1)\Delta)\\ &\leqslant\ 1-\exp\left\{-2a^{-1}(e^{a\Delta}-1)e^{am\Delta}\right\}\end{aligned}\tag{89}$$

which decays exponentially as $m\to-\infty$ thus yielding a summable upper bound.

Equation (80) is a consequence of Eq. (79) but can also be obtained directly from the degenerate distribution (74): Choose any $\epsilon>0$.

$$\begin{aligned}&\left(1-a^{-1}-\epsilon\right)P_{N,a_N}\left(\frac{n_0}{N}\geqslant 1-a^{-1}-\epsilon\right)\\ &\quad\leqslant\frac{\langle n_0\rangle_{N,a_N}}{N}\leqslant\left(1-a^{-1}+\epsilon\right)P_{N,a_N}\left(\frac{n_0}{N}\leqslant 1-a^{-1}+\epsilon\right)\\ &\qquad+P_{N,a_N}\left(\frac{n_0}{N}>1-a^{-1}+\epsilon\right).\end{aligned}\tag{90}$$

Taking the limit $N\to\infty$ and applying Eq. (74),

$$1-a^{-1}-\epsilon\leqslant\liminf\frac{\langle n_0\rangle_{N,a_N}}{N}\leqslant\limsup\frac{\langle n_0\rangle_{N,a_N}}{N}\leqslant 1-a^{-1}+\epsilon\tag{91}$$

which holds for any $\epsilon>0$ and thus for $\epsilon=0$ as well.
III. If Z_{N,a_N} denotes the N-particle partition function then

$$F_N^0=-\frac{1}{\beta}\ln Z_{N,a_N}=-\frac{1}{\beta}\left(-\frac{1}{2}Na_N+\ln Q_{N,a_N}\right).\tag{92}$$

From Eq. (43)

$$\ln Q_{N,a_N}=-\sum_{k=1}^{N}\ln(1-e^{-ka_N})\tag{93}$$

so that

$$\lim_{N\to\infty} \gamma_N \ln Q_{N,a_N} = -\lim_{N\to\infty} \sum_{k=1}^{N} \gamma_N \ln(1-e^{-ak\gamma_N})$$
$$= -\int_0^\infty \ln(1-e^{-ax})\,\mathrm{d}x \tag{94}$$

because $\gamma_N \to 0$ and $N\gamma_N = \ln N \to \infty$. Expanding the logarithm and integrating term by term ($\sum_{k=1}^{n} e^{-kax}/k \to -\ln(1-e^{-ax})$ monotonically) we find $\pi^2/6a$ and, hence, Eq. (81). ▮

Remarks . (1) Equation (76) implies $|n_0 - \langle n_0\rangle_{N,a_N}| \sim N/\ln N$ for $a>1$. These are huge fluctuations even compared with the super-normal fluctuations $|n_0 - \langle n_0\rangle| \sim N^{2/3}$ in the condensation regime of the three-dimensional homogenous Bose gas.[28]
(2) There is no singularity in $\lim \gamma_N F_N^0 = -\pi^2/6\beta a$ at the critical point $a=1$.
(3) For $\gamma_N = 1/N$ Eq. (94) yields an extensive free energy

$$\lim_{N\to\infty} \frac{1}{N} F_N^0 = -\beta^{-1} \lim_{N\to\infty} \frac{1}{N} \ln Q_{N,a_N} = \beta^{-1} \int_0^1 \ln(1-e^{-ax})\,\mathrm{d}x \tag{95}$$

so that this scaling corresponds to the homogenous limit. According to Corollary 3.1, $T_c=0$ in this case as it has to be in the one dimensional homogenous Bose gas.

3.2. Generalized Bose Condensation at All Temperatures in the Noninteracting Gas

In the three-dimensional homogenous noninteracting Bose gas above the critical temperature the mean occupation number of each one-particle state remains finite in the thermodynamic limit. This is immediately seen in the grand-canonical ensemble and is valid in the canonical ensemble as well, due to the strong equivalence of ensembles (the Kac density is a Dirac delta). The situation is quite different in the one-dimensional scaled harmonic trap when $\gamma_N = \ln N/N$. As we shall see, there is a complete GBEC at all temperatures. It will also be shown that below the critical temperature the condensate is not fragmented, $\langle n_i\rangle_{N,a_N} = o(N)$ for $i>0$. This means that for $a>1$ a condensate in the ground state of H^0 whose density is $1-1/a$ coexists with a generalized condensate of density $1/a$.

The intuition behind the results of this section is guided by the following observation. (We drop the subscript a_N which plays no role here.)

Lemma 3.2. For any i

$$\langle n_i \rangle_N = \sum_{M=0}^{N} \langle n_0 \rangle_{N-M} \; P_N \left(\sum_{j=0}^{i-1} n_j = M \right). \tag{96}$$

Proof. The reader can easily check that because of

$$\varepsilon_n = \varepsilon_0 + n(\varepsilon_1 - \varepsilon_0)$$

the conditional distribution of n_i, given the number of particles in the lower lying eigenstates, satisfies

$$P_N \left(n_i = m \left| \sum_{j=0}^{i-1} n_j = M \right. \right) = P_{N-M}(n_0 = m) \; . \tag{97}$$

Multiplying by m and summing over it the right member becomes $\langle n_0 \rangle_{N-M}$ while on the left-hand side we obtain the conditional expectation value $\langle n_i | n_0 + \cdots + n_{i-1} = M \rangle_N$. Multiplying by the probability of the condition and summing over M yields Eq. (96). ∎

Because for $a < 1$ $n_0 \approx N^a$, we expect that

$$\langle n_i \rangle_{N,a_N} \asymp \langle n_0 \rangle_{N - iN^a, a_N} \asymp N^a \tag{98}$$

if $i \ll N^{1-a}$, and that an $o(N)$ number of lowest lying levels carry roughly all the particles. This should hold true for $a > 1$ as well and, because of $n_0 \approx N(1 - 1/a)$, we expect also that

$$\langle n_i \rangle_{N,a_N} \asymp \langle n_0 \rangle_{N/a - (i-1)a^{-1} N \ln\ln N / \ln N, a_N} \asymp a^{-1} N \ln\ln N / \ln N \tag{99}$$

for $1 \leqslant i \ll \ln N / \ln\ln N$. Not all these conjectures will be verified below. We start by proving the second part of Eq. (98).

Proposition 3.4. All the results of Theorem 3 remain valid if the scaling $a_N = a \ln N / N$ is replaced by $a'_N = (1 + \eta_N) a_N$ where $\eta_N = o(1/\ln N)$.

Proof. Consider A_{Nk}, defined in Eq. (53), as a function of a_N and λ_N. If a_N is replaced by a'_N given above, one finds that for any sequence λ_N

$$A_{Nk}(a_N, \lambda_N) / A_{Nk}(a'_N, \lambda_N) \to 1$$

at least as fast as $\eta_N \ln N$ tends to zero. Therefore Eq. (68) can be replaced by

$$P_{N,a'_N}(\frac{n_0}{N} \geqslant \lambda_N) = e^{-(1+\zeta_N)A_{N1}(a_N,\lambda_N)} \tag{100}$$

where $\zeta_N \to 0$. Because the results of Theorem 3 grouped under points I and II were derived from Eq. (68) and did not depend on the particular way ϵ_N tended to zero, the modified scaling will provide the same outcome. Although we do not use it, we note that the free energy also will be the same. ▮

As a corollary we obtain

Corollary 3.3. Let $S = S(N) = o(N/\ln N)$. For $a < 1$

$$\lim_{N\to\infty} N^{-a} \langle n_0 \rangle_{N+S,a_N} = 1. \tag{101}$$

Proof. Because $\lim(N+S)^a/N^a = 1$,

$$\lim_{N\to\infty} N^{-a} \langle n_0 \rangle_{N+S,a_N} = \lim_{N\to\infty} N^{-a} \langle n_0 \rangle_{N,a'_N}$$

with

$$a'_N = \frac{1 + (\ln N)^{-1} \ln(1 - S/N)}{1 - S/N} \; a_N \equiv (1 + \eta_N) a_N. \tag{102}$$

One can easily verify that $\eta_N \ln N \to 0$. Thus, the result follows by applying Proposition 3.4. ▮

Lemma 3.3. For any noninteracting Bose gas with a one-particle spectrum $\{\varepsilon_j\}$ the average occupation numbers in the N-particle canonical ensemble are strictly decreasing with the energy,

$$\langle n_j \rangle_N < \langle n_i \rangle_N \qquad \text{if} \qquad \varepsilon_i < \varepsilon_j. \tag{103}$$

Proof. Let $x_i = \exp(-\beta\varepsilon_i)$ and let Z_N and $Z_{N,i,j}$ denote the N-particle partition functions of the full system and of the system with missing levels i and j, respectively.

$$
\begin{aligned}
&Z_N[\langle n_i\rangle_N - \langle n_j\rangle_N] \\
&\quad = \sum_{l,m}(m-l)x_i^m x_j^l Z_{N-l-m,i,j} \\
&\quad = \sum_{0\leqslant l<m\leqslant N}(m-l)[x_i^m x_j^l - x_i^l x_j^m]Z_{N-l-m,i,j} \qquad (104)\\
&\quad = \sum_{0\leqslant l<m\leqslant N}(m-l)(x_i x_j)^l[x_i^{m-l} - x_j^{m-l}]Z_{N-l-m,i,j} > 0. \quad \blacksquare \qquad (105)
\end{aligned}
$$

Theorem 4. For any $a \equiv \hbar\omega\beta > 0$ the scaling $a_N = a \ln N/N$ leads to a complete generalized Bose-Einstein condensation. For $a > a_c = 1$ there is no fragmented condensation but the generalized condensate on the levels $1, 2, \ldots$ is at its critical point. In particular:
(i) For any $a > 0$

$$
\lim_{N\to\infty} \frac{1}{N} \sum_{j<2/a_N} \langle n_j\rangle_{N,a_N} = 1. \tag{106}
$$

(ii) For $a < 1$

$$
\lim_{N\to\infty} N^{-a}\langle n_i\rangle_{N,a_N} = 1 \quad \text{if} \quad i = o\left(\frac{N^{1-a}}{\ln^2 N}\right). \tag{107}
$$

(iii) For $a > 1$

$$
\lim_{N\to\infty} N^{-1}\langle n_i\rangle_{N,a_N} = 0 \quad \text{if} \quad i \geqslant 1 \tag{108}
$$

but

$$
\lim_{N\to\infty} N^{-\eta}\langle n_1\rangle_{N,a_N} = \infty \quad \text{for any} \quad \eta < 1. \tag{109}
$$

Proof. (i) Let $J = 2/a_N$. We shall prove that

$$
\ln P_{N,a_N}\left(\sum_{j\geqslant J} n_j = m\right) \leqslant -\left(2 - \frac{\pi^2}{12}\right)m \quad \text{if} \quad m \geqslant J. \tag{110}
$$

Then for $b = e^{-2+\pi^2/12}$

$$\sum_{m \geqslant J} m P_{N,a_N}\left(\sum_{j \geqslant J} n_j = m\right) \leqslant \sum_{J \leqslant m \leqslant N} m b^m < \left[\frac{J}{1-b} + \frac{b}{(1-b)^2}\right] b^J \to 0 \tag{111}$$

as N tends to infinity. Thus for N large enough

$$0 < N - \sum_{j<J} \langle n_j \rangle_{N,a_N} = \sum_{j \geqslant J} \langle n_j \rangle_{N,a_N} = \sum_{m<J} m P_{N,a_N}\left(\sum_{j \geqslant J} n_j = m\right) + \sum_{m \geqslant J} m P_{N,a_N}\left(\sum_{j \geqslant J} n_j = m\right) < J+1. \tag{112}$$

Dividing by N and taking the limit we obtain Eq. (106).

To prove Eq. (110), consider the probability on the left side. With the notation $x = e^{-a_N}$

$$P_{N,a_N}\left(\sum_{j \geqslant J} n_j = m\right) = Q^{-1}_{N,a_N}\left(\sum_{\{n_j\}_{j \geqslant J}:\sum n_j = m} x^{\sum_{j \geqslant J} j n_j}\right) \times \left(\sum_{\{n_j\}_{j=1}^{J-1}:\sum n_j \leqslant N-m} x^{\sum_{j=1}^{J-1} j n_j}\right). \tag{113}$$

The quantity in the first bracket is $x^{mJ} Q_{m,a_N}$, cf. Eq. (41). The quantity in the second bracket can be bounded by dropping the constraint $\sum n_j \leqslant N - m$,

$$\left(\sum_{\{n_j\}_{j=1}^{J-1}:\sum n_j \leqslant N-m} x^{\sum_{j=1}^{J-1} j n_j}\right) < \prod_{j=1}^{J-1} \frac{1}{1-x^j} = Q_{J-1,a_N}, \tag{114}$$

see Eq. (43). Thus,

$$P_{N,a_N}\left(\sum_{j \geqslant J} n_j = m\right) < x^{mJ} Q^{-1}_{N,a_N} Q_{m,a_N} Q_{J-1,a_N} \leqslant x^{mJ} Q_{J-1,a_N} \tag{115}$$

because Q_{m,a_N} is increasing with m. Taking the logarithm and using $Ja_N = 2$,

$$\ln P_{N,a_N}\left(\sum_{j\geqslant J} n_j = m\right) < -2m - \sum_{j=1}^{J-1} \ln(1-x^j). \tag{116}$$

Now $-\ln(1-x^j)$ is a positive decreasing function of j, therefore

$$\begin{aligned}
&-\sum_{j=1}^{J-1} \ln(1-e^{-ja_N}) \\
&\quad\leqslant -\frac{1}{a_N}\int_0^2 \ln(1-e^{-t})\,\mathrm{d}t \\
&\quad< -\frac{J}{2}\int_0^\infty \ln(1-e^{-t})\,\mathrm{d}t = J\pi^2/12 \leqslant m\pi^2/12
\end{aligned} \tag{117}$$

for $m\geqslant J$. Sustituting this into Eq. (116) we obtain Eq. (110).
(ii) To prove Eq. (107) we remark that because of

$$\langle n_i\rangle_{N,a_N} \leqslant \langle n_0\rangle_{N,a_N} \asymp N^a,$$

cf. Lemma 3.3, it suffices to show that $\lim N^{-a}\langle n_i\rangle_{N,a_N} \geqslant 1$. Let $K = i\ln N$; then $K = o(N^{1-a}/\ln N)$. From Eq. (96)

$$\begin{aligned}
\langle n_i\rangle_{N,a_N} &\geqslant \sum_{M=0}^{KN^a} \langle n_0\rangle_{N-M,a_N} P_{N,a_N}\left(\sum_{j=0}^{i-1} n_j = M\right) \\
&\geqslant \langle n_0\rangle_{N-\lfloor KN^a\rfloor,a_N} P_{N,a_N}\left(\sum_{j=0}^{i-1} n_j \leqslant KN^a\right)
\end{aligned} \tag{118}$$

where we have used the monotonic increase of $\langle n_0\rangle_{k,a_N}$ with k, derived in ref. 29. By Corollary 3.3 Eq. (118) implies

$$\lim_{N\to\infty} N^{-a}\langle n_i\rangle_{N,a_N} \geqslant 1 - \lim_{N\to\infty} P_{N,a_N}\left(\sum_{j=0}^{i-1} n_j \geqslant KN^a\right). \tag{119}$$

We show that the limit on the right-hand side is zero.

$$\begin{aligned}
P_{N,a_N}\left(\sum_{j=0}^{i-1} n_j \geqslant L\right) &= P_{N,a_N}\left(\frac{1}{i}\sum_{j=0}^{i-1} n_j \geqslant \frac{L}{i}\right) \\
&\leqslant P_{N,a_N}\left(\max\{n_0,\ldots,n_{i-1}\} \geqslant \frac{L}{i}\right) \\
&\leqslant \sum_{j=0}^{i-1} P_{N,a_N}\left(n_j \geqslant \frac{L}{i}\right) \\
&= \sum_{j=0}^{i-1} Q_{N,a_N}^{-1} \sum_{m \geqslant L/i} x^{mj} \sum_{\{n_k\}_{k\neq j}:\sum n_k = N-m} x^{\sum k n_k} \\
&\leqslant \sum_{j=0}^{i-1} x^{Lj/i} P_{N,a_N}(n_0 \geqslant L/i, n_j = 0) \\
&\leqslant P_{N,a_N}(n_0 \geqslant L/i) \sum_{j=0}^{i-1} x^{Lj/i}. \qquad (120)
\end{aligned}$$

For $L = KN^a = iN^a \ln N$ we can apply Eq. (72) with $\lambda_N = \ln N/N^{1-a}$. We find

$$\begin{aligned}
P_{N,a_N}\left(\sum_{j=0}^{i-1} n_j \geqslant KN^a\right) &\leqslant \exp\left\{-\frac{1}{N^a}\frac{e^{a\ln^2 N/N^{1-a}}-1}{e^{a\ln N/N}-1}(1+\epsilon_N)\right\}\sum_{j=0}^{i-1} x^{Lj/i} \\
&< N^{-(1+\epsilon_N)} i \to 0 \qquad (121)
\end{aligned}$$

which proves the assertion.
(iii) Choose any δ such that $1 < \delta < a$. Let $K = (\delta/a)N$ and for the sake of notational simplicity suppose that K is integer. From Eq. (96)

$$\langle n_1 \rangle_{N,a_N} \leqslant \langle n_0 \rangle_{K,a_N} P_{N,a_N}(n_0 \geqslant N-K) + N P_{N,a_N}(n_0 < N-K). \quad (122)$$

Now $N = (a/\delta)K$, thus

$$a_N = \delta\ \frac{\ln K + \ln(a/\delta)}{K} < (2\delta - 1)\frac{\ln K}{K}$$

if N is large enough. Because $\langle n_0\rangle_{K,a}$ is an increasing function of a (or of β, see[29]),

$$\begin{aligned}
0 &\leqslant \lim_{N\to\infty}\frac{1}{N}\langle n_1\rangle_{N,a_N}\\
&\leqslant \frac{\delta}{a}\left[1-\frac{1}{2\delta-1}\right]+\lim_{N\to\infty}P_{N,a_N}(n_0<N-K)\\
&= \frac{\delta}{a}\left[1-\frac{1}{2\delta-1}\right]+1-\lim_{N\to\infty}P_{N,a_N}\left(\frac{n_0}{N}\geqslant 1-\frac{\delta}{a}\right)\\
&= \frac{\delta}{a}\left[1-\frac{1}{2\delta-1}\right] \qquad (123)
\end{aligned}$$

where we have used Eqs. (80) and (74). Because this holds for any $\delta>1$, letting δ tend to 1 we obtain Eq. (108) for $i=1$. For $i>1$, Eq. (108) results by applying Lemma 3.3.

To prove Eq. (109) choose any $\eta<1$. From Eq. (96)

$$\begin{aligned}
\langle n_1\rangle_{N,a_N} &\geqslant \sum_{M>K}\langle n_0\rangle_{M,a_N}P_{N,a_N}(n_0=N-M)\\
&\geqslant \langle n_0\rangle_{K,a_N}[1-P_{N,a_N}(n_0\geqslant N-K)]. \qquad (124)
\end{aligned}$$

Let $\delta=\frac{1}{2}(\eta+1)$, then $\delta<1$. Define $K=(\delta/a)N$. By Eq. (74) now

$$\lim_{N\to\infty}P_{N,a_N}(n_0\geqslant N-K)=\lim_{N\to\infty}P_{N,a_N}(n_0/N\geqslant 1-\delta/a)=0.$$

Because $a_N>\delta\ln K/K$, with Eq. (78) we obtain

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N^\eta}\langle n_1\rangle_{N,a_N} &\geqslant \left(\frac{\delta}{a}\right)^\eta\lim_{K\to\infty}K^{1-\delta}\frac{1}{K^\delta}\langle n_0\rangle_{K,\delta\ln K/K}\\
&= \left(\frac{\delta}{a}\right)^\eta\lim_{K\to\infty}K^{1-\delta}=\infty \qquad (125)
\end{aligned}$$

which finishes the proof of the theorem. We note that with more effort, using Eqs. (76) and (79) one could prove the precise asymptotics (99) for, at least, any fixed $i\geqslant 1$. ▮

3.3. Generalized Bose Condensation at all Temperatures in the Interacting Gas

In this section we investigate the possibility of introducing a nontrivial interaction in the system without loosing Bose-Einstein condensation. By nontrivial we mean an interaction providing an energy contribution comparable with or larger than the free energy of the free system. The Hamiltonian of this latter is (50) with $\gamma_N = \ln N/N$ while that of the interacting gas is

$$H_N = H_N^0 + U_N = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x_i^2} + \gamma_N V(\sqrt{\gamma_N} x_i) \right] + U_N(x_1, \dots, x_N). \tag{126}$$

We return to the notations introduced in Section 2.1. Mean values with respect to the noninteracting gas in the scaled harmonic trap will thus be denoted by $\langle \cdot \rangle_{\beta H_N^0}$, in contrast to $\langle \cdot \rangle_{N, a_N}$ used in Sections 3.1 and 3.2. Also, in the case of the spectrally deformed noninteracting gas we apply the notation $\langle \cdot \rangle_{\beta H_N^0(J,\delta)}$.

Theorem 5. Interacting bosons in a scaled harmonic trap with

$$\gamma_N = \frac{\ln N}{N} \quad \text{and} \quad U_N \geqslant -BN, \ \langle U_N \rangle_{\beta H_N^0} = o(N \ln N)$$

undergo a complete generalized Bose-Einstein condensation at all temperatures: For any $\beta > 0$

$$\lim_{s \to 0} \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{sN} \langle n_j \rangle_{\beta H_N} = 1. \tag{127}$$

For stable pair interactions,

$$U_N(x_1, \dots, x_N) = \sum_{i<j} u_N(x_i - x_j),$$

$\langle U_N \rangle_{\beta H_N^0} = o(N \ln N)$ and thus (127) holds true if $\|u_N\|_1 = o(\gamma_N)$. If $u_N(x) = b_N u(\alpha_N x)$, the condition reads $\|u\|_1 < \infty$ and $b_N = o(\alpha_N \ln N/N)$.

Observe that the scaling condition $\|u_N\|_1 = o(\ln N/N)$ is somewhat weaker than Eq. (3) and, again, examples from mean-field type to sharply concentrated interactions can be obtained.

We note that the discussion of Section 2.2 applies to the case of a fixed harmonic potential. In analogy with Theorem 2 we find that for a *fixed* harmonic trap and scaled interactions satisfying $\langle U_N\rangle_{\beta H_N^0} = o(N^2)$ there is a complete GBEC at all $\beta > 0$. In Theorem 5 we conclude that for somewhat weaker interactions the same holds true even if the frequency of the confining harmonic potential is scaled with $\gamma_N = \ln N/N$. One could also show without much further ado that for more general scalings $\omega_N = \gamma_N\omega$, where $\gamma_N \geqslant \ln N/N$, and $\langle U_N\rangle_{\beta H_N^0} = o(N^2\gamma_N)$ there is a complete GBEC at all temperatures.

Proof. We use Lemma 2.1 and Proposition 2.1. If $J = \lfloor sN \rfloor$ then $\delta = \varepsilon_J = \hbar\omega\gamma_N(\lfloor sN\rfloor + \frac{1}{2}) \approx s\hbar\omega \ln N$ implying

$$\lim \frac{1}{N\delta}[\langle U_N\rangle_{\beta H_N^0} - \inf U_N] = 0$$

and therefore

$$\lim \frac{1}{N}\sum_{j=0}^{J}\langle n_j\rangle_{\beta H_N} \geqslant \lim \frac{1}{N}\sum_{j=0}^{J}\langle n_j\rangle_{\beta H_N^0(J,\delta)}.$$

We can conclude by showing that

$$\langle n_j\rangle_{\beta H_N^0(J,\delta)} \geqslant \langle n_j\rangle_{\frac{1}{2}\beta H_N^0} \tag{128}$$

because from Theorem 4 it follows that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=0}^{J}\langle n_j\rangle_{\frac{1}{2}\beta H_N^0} = 1$$

for any β. To prove Eq. (128) notice that the spectrum of $H^0(J,\delta) - \varepsilon_0 - \varepsilon_J$ is a part of the spectrum of $\frac{1}{2}(H^0 - \varepsilon_0)$. Indeed,

$$\text{spec}\left\{\frac{1}{2}(H^0 - \varepsilon_0)\right\} = \frac{1}{2}\gamma_N\hbar\omega \times \{0, 1, 2, \ldots\} \tag{129}$$

while

$$\begin{aligned}\text{spec}&\left\{H^0(J,\delta) - \varepsilon_0 - \varepsilon_J\right\}\\ &= \frac{1}{2}\gamma_N\hbar\omega \times \{0, 1, 2, \ldots, 2J, 2J+1, 2J+3, 2J+5, \ldots\}.\end{aligned} \tag{130}$$

In ref. 29, we proved the following result. ▮

Lemma 3.4. Let H^0 and H^1 be two one-particle Hamiltonians, both $e^{-\beta H^0}$ and $e^{-\beta H^1}$ trace class and

$$\mathrm{spec}\, H^0 \subset \mathrm{spec}\, H^1$$

where repeated eigenvalues are considered separately. If ε_i is a common eigenvalue then the averages of the occupation number n_i in the two non-interacting ensembles satisfy the inequality

$$\langle n_i \rangle_{\beta H_N^1} < \langle n_i \rangle_{\beta H_N^0}. \tag{131}$$

On the left- and right-hand sides of (131) n_i denotes $n[\varphi_i^1]$ and $n[\varphi_i^0]$, respectively, where $H^0\varphi_i^0 = \varepsilon_i \varphi_i^0$, $H^1\varphi_i^1 = \varepsilon_i \varphi_i^1$, and φ_i^0 and φ_i^1 may be different. If η_j are the eigenvalues of H^1 not contained in the spectrum of H^0 and $x_j = e^{-\beta\eta_j}$ then

$$\begin{aligned} \langle n_i \rangle_{\beta H_N^1} &= \frac{1}{Z[\beta H_N^1]} \sum_{m=0}^{N} \langle n_i \rangle_{\beta H_{N-m}^0} Z[\beta H_{N-m}^0] \sum_{\{k_j\}:\sum k_j = m} \prod_j x_j^{k_j} \\ &\equiv \sum_{m=0}^{N} \langle n_i \rangle_{\beta H_{N-m}^0} P_m \end{aligned} \tag{132}$$

where Z denotes partition functions. Because $\sum_{m=0}^{N} P_m = 1$, Eq. (131) follows from the strict monotonic increase of $\langle n_i \rangle_{\beta H_N^0}$ with N, proven in ref. 29. We obtain Eq. (128) and thereby the proof of a complete GBEC by replacing H^0 with $H^0(J,\delta) - \varepsilon_0 - \varepsilon_J$ and H^1 with $\frac{1}{2}(H^0 - \varepsilon_0)$ in Lemma 3.4. (To avoid confusion we precise that on the right-hand side of equation (132) H_{N-m}^0 is defined with the same one-particle spectrum for each m.)

In the case of pair interactions

$$\langle u_N \rangle_{\beta H_N^0} = \frac{2}{N(N-1)} \langle U_N \rangle_{\beta H_N^0} \tag{133}$$

and to satisfy the condition of the theorem we need $\langle u_N \rangle_{\beta H_N^0} = o(\ln N/N)$. We show below that

$$|\langle u_N \rangle_{\beta H_N^0}| \leqslant \frac{\sqrt{2}}{\lambda_\beta}(1 + O(\sqrt{\gamma_N}))\|u_N\|_1 \tag{134}$$

and therefore $\|u_N\|_1 = o(\gamma_N)$ is a sufficient condition for GBEC to hold for any $\beta > 0$. Now

$$\begin{aligned}\langle u_N\rangle_{\beta H_N^0} &= Z[\beta H_N^0]^{-1}\sum_{g\in S_N}\int \mathrm{d}x_1\,\mathrm{d}x_2 u_N(x_1-x_2)\\ &\quad\times\int \mathrm{d}x_3\ldots\mathrm{d}x_N\prod_{j=1}^{N}\langle x_j|e^{-\beta H^0}|x_{g(j)}\rangle\\ &= \sum_{g\in S_N}\frac{\mathrm{Tr}\,U(g)e^{-\beta H_N^0}}{\sum_{h\in S_N}\mathrm{Tr}\,U(h)e^{-\beta H_N^0}}\\ &\quad\times\int \mathrm{d}x_1\,\mathrm{d}x_2 u_N(x_1-x_2)\mu_{g,\beta}(x_1,x_2)\end{aligned}\tag{135}$$

where S_N is the group of the permutations of $1,2,\ldots,N$, $U(g)$ is the unitary representation of the permutation g in the full N-particle Hilbert space and Tr is the trace in this space. In the average above there are two kinds of probability measures $\mu_{g,\beta}$. If 1 and 2 are in different cycles of g then

$$\mu_{g,\beta}(x,y)=\mu_{\ell_1(g)\beta}(x)\mu_{\ell_2(g)\beta}(y)\equiv\mu_{\beta_1}(x)\mu_{\beta_2}(y)\tag{136}$$

where $\ell_i(g)$ is the length of the cycle of g containing i and

$$\mu_\beta(x)=\frac{\langle x|e^{-\beta H^0}|x\rangle}{\mathrm{tr}\,e^{-\beta H^0}}.\tag{137}$$

If 1 and 2 are in the same cycle of length ℓ and $g^j(1)=2$ then

$$\mu_{g,\beta}(x,y)=\frac{\langle x|e^{-j\beta H^0}|y\rangle\langle y|e^{-(\ell-j)\beta H^0}|x\rangle}{\mathrm{tr}\,e^{-\ell\beta H^0}}.\tag{138}$$

In the first case by Fourier transforming and applying Schwarz inequality and Parseval formula we get

$$\left|\int \mathrm{d}x_1\,\mathrm{d}x_2 u_N(x_1-x_2)\mu_{g,\beta}(x_1,x_2)\right|\leqslant\|u_N\|_1\|\mu_{\beta_1}\|\ \|\mu_{\beta_2}\|\tag{139}$$

with $\|\mu_\beta\|$ denoting the usual L^2 norm of μ_β. In the second case

$$\begin{aligned}\left|\int \mathrm{d}x_1\,\mathrm{d}x_2 u_N(x_1-x_2)\mu_{g,\beta}(x_1,x_2)\right| &\leqslant \int \mathrm{d}z|u_N(z)|\int \mathrm{d}y\,\mu_{g,\beta}(z+y,y)\\ &\leqslant \|u_N\|_1\sup_z\int \mathrm{d}y\,\mu_{g,\beta}(z+y,y).\end{aligned}\tag{140}$$

At this point we recall that Eq. (6) involves a transformation between two Hilbert spaces, that of L^2 functions of the variables $\mathbf{x}_i$ and $\mathbf{y}_i = \alpha_N \mathbf{x}_i$, respectively. Previously we have only been interested in properties depending on the spectrum of βH^0. Because β and ω appeared in a single dimensionless combination $a = \hbar\omega\beta$, γ_N could be considered to multiply a and yield $a_N = a\gamma_N$. Here we are estimating functions and have to decide which space to work in. In accordance with Eqs. (50) and (126) we choose to scale the potential which amounts to replace ω by $\omega_N = \omega\gamma_N$ and a by $a_N = \hbar\omega_N\beta$ and to keep β as an independent unscaled variable.

We use Mehler formula[30]

$$\langle x|e^{-\beta H^0}|y\rangle = \left[\frac{m\omega_N}{2\pi\hbar\sinh a_N}\right]^{\frac{1}{2}} \exp\left\{-\frac{m\omega_N(x^2+y^2)}{2\hbar\tanh a_N} + \frac{m\omega_N xy}{\hbar\sinh a_N}\right\} \tag{141}$$

to obtain

$$\mu_\beta(x) = \sqrt{s/\pi}\exp(-sx^2) \qquad s = s(\beta) = \frac{m\omega_N}{\hbar}\tanh\hbar\omega_N\beta/2. \tag{142}$$

Because

$$\frac{\partial\mu_\beta(x)}{\partial s} = \mu_\beta(x)[(2s)^{-1} - x^2] = \mu_\beta(x)[\langle x^2\rangle_{\mu_\beta} - x^2], \tag{143}$$

$\|\mu_\beta\|$ is an increasing function of s and, therefore, of β:

$$\begin{aligned}\frac{\partial}{\partial s}\int\mu_\beta^2\,\mathrm{d}x &= 2\int_{x^2<1/2s}\mu_\beta(x)\frac{\partial\mu_\beta}{\partial s}(x)\,\mathrm{d}x + 2\int_{x^2>1/2s}\mu_\beta(x)\frac{\partial\mu_\beta}{\partial s}(x)\,\mathrm{d}x \\ &> 2\mu_\beta(\sqrt{1/2s})\left[\int_{x^2<1/2s}\frac{\partial\mu_\beta}{\partial s}(x)\,\mathrm{d}x + \int_{x^2>1/2s}\frac{\partial\mu_\beta}{\partial s}(x)\,\mathrm{d}x\right] = 0.\end{aligned}$$

Here we used that $\mu_\beta(x)$ is a decreasing function of $|x|$ and $\int\mu_\beta\,\mathrm{d}x = 1$. Thus, we can bound (139) by inserting the largest possible β_1 and β_2 on the right-hand side. Because

$$\max_g \ell_i(g)\beta = (N-1)\beta < N\beta,$$

we find

$$\|\mu_{\beta_1}\| \; \|\mu_{\beta_2}\| \; < \; \|\mu_{N\beta}\|^2 = \frac{s(N\beta)}{\pi} \int e^{-2s(N\beta)x^2} \, \mathrm{d}x = \sqrt{s(N\beta)/2\pi}$$
$$= \left[\frac{m\omega_N}{2\pi\hbar} \tanh \hbar\omega_N N\beta/2\right]^{1/2}$$
$$< \left[\frac{m\omega_N}{2\pi\hbar}\right]^{1/2} = O(\sqrt{\gamma_N}). \tag{144}$$

To estimate the right-hand side of Eq. (140), observe that

$$\nu(z) = \int \mu_{g,\beta}(z+y, y) \, \mathrm{d}y$$

is normalized. One then obtains

$$\nu(z) = \sqrt{D/\pi} e^{-Dz^2}$$
$$D = \frac{m\omega_N}{4\hbar} \left(\frac{1}{\tanh ja_N} + \frac{1}{\tanh(\ell - j)a_N} + \frac{1}{\sinh ja_N} + \frac{1}{\sinh(\ell - j)a_N} \right) \tag{145}$$

and thus $\sup_z \nu(z) = \nu(0) = \sqrt{D/\pi}$. Now D attains its maximum for $\ell = 2$,

$$\max_{\ell \geqslant 2} D = \frac{m\omega_N}{2\hbar} \left(\frac{1}{\tanh a_N} + \frac{1}{\sinh a_N} \right) = \frac{m}{\hbar^2 \beta} + O(\gamma_N^2) = \frac{2\pi}{\lambda_\beta^2} + O(\gamma_N^2).$$

Thus for the permutations containing 1 and 2 in the same cycle we have the uniform upper bound

$$\sup_z \int \mu_{g,\beta}(z+y, y) \, \mathrm{d}y \leqslant \frac{\sqrt{2}}{\lambda_\beta} + O(\gamma_N^2). \tag{146}$$

Finally, we obtain the inequality (134) by substituting (144) and (146) into (139) and (140), respectively, and using these latter to bound the integrals in (135). The consequence $b_N = o(\alpha_N \gamma_N)$ for scaled pair interactions is trivial. By this we finished the proof of the theorem.

4. SUMMARY

This paper is an extension of our earlier study[14] of Bose-Einstein condensation of trapped Bose gases. We have concentrated on two problems. The first, treated in Section 2, concerns bosons in a fixed trap interacting with unscaled interactions; that is, neither the confining potential nor the pair interaction between two particles depends on the number of particles. Harmonic or weaker trap potentials combined with unscaled interactions are too difficult to deal with. However, if the confining potential increases faster than quadratically at infinity, in one dimension we can prove a complete generalized Bose-Einstein condensation at all temperatures. This means that as N tends to infinity all but a vanishing fraction of particles will be distributed over a set of one-particle states whose number is asymptotically negligible compared with N. The result does not provide a more precise information about the number of one-particle states carrying the condensate. The generalized condensation may eventually prove to be normal and non-fragmented.

The second problem we have investigated in this paper is the condensation of bosons in a one-dimensional scaled harmonic trap. In a previous work[25] it was shown that for a particular scaling, when the oscillator frequency ω is replaced by $\omega \ln N/N$, Bose-Einstein condensation occurs with a phase transition in the noninteracting gas. Sections 3.1 and 3.2 have been devoted to a detailed study of this system. Our most interesting finding is that the occupation of the excited states is anomalously large and results in a complete generalized Bose-Einstein condensation at all temperatures, superimposed on the normal Bose-condensation below the critical temperature. In Section 3.3 we have studied the same system in the case when there is also a suitably scaled interaction among the particles. We have shown that the complete GBEC is preserved by the interaction without being able to prove that the phase transition also persists.

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